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Isometries of Riemannian Images*

by

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Abstract. Let I be the family of all isometries from a metric space M onto M. Let the distance of p and q on a covering surface C over M be the infimum of the lengths of curves connecting p and q on C. Is an isometry from a covering surface C_1 onto another C_2 over M induced in a natural manner by a member of I? Three, function-theoretic, positive answers will be proposed.

1. Introduction

All functions are assumed to be nonconstant. The Riemannian image $R(f)$ of a function f meromorphic in a domain D in the complex plane covers the Riemann sphere Σ. The distance $d(f(z), f(w))$ on $R(f)$ is then defined as the infimum of the lengths of all curves connecting $f(z)$ and $f(w)$ on $R(f)$. Let g be meromorphic in D, and suppose that there is an isometry from $R(f)$ onto $R(g)$. We shall show that this isometry is either a rotation of Σ or else a rotation followed by the reflexion with respect to the great circle corresponding to the real axis.

Analogous results will be observed on replacing the distance by the two natural ones when f and g are pole-free or bounded by one at the same time.

2. Results

Henceforth j stands for 1, 2, or 3. Let $S_1=\{|z|\leqq\infty\}$ or Σ, $S_2=\{|z|<\infty\}$, and $S_3=\{|z|<1\}$. Here, Σ is of diameter one touching S_2 at the origin. Then, S_j is endowed with the distance d_j, namely,

$$\begin{aligned} d_j(z, w) &= \tan^{-1}(|z-w|/|1+\bar{z}w|) && \text{if } j=1\,; \\ &= |z-w| && \text{if } j=2\,; \\ &= \tanh^{-1}(|z-w|/|1-\bar{z}w|) && \text{if } j=3\,, \end{aligned}$$

where $0\leqq\tan^{-1}x\leqq\pi/2$ for $0\leqq x\leqq+\infty$. Let T_j be the family of all functions,

$$a(z-b)/\{1+(2-j)\bar{b}z\}\,, \quad \text{and} \quad a(\bar{z}-b)/\{1+(2-j)\bar{b}\bar{z}\}\,,$$

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where $|a|=1$, $b \in S_j$; read a/z or $a/\bar{z}$ if $b=\infty$. Then, each $\phi_j \in T_j$ preserves d_j. Let $D \subset S_2$ be a domain, let f_j: $D \to S_j$ be meromorphic, and let $R(f_j)$ be the Riemannian image of D by f_j covering S_j. Set

$$\delta f_j(z) = |f_j'(z)| / \{1+(2-j)|f_j(z)|^2\}\,, \qquad z \in D\,;$$

if $f_1(z)=\infty$, then $\delta f_1(z) = |(1/f_1)'(z)|$. With the distance

$$d(f_j(z), f_j(w); f_j) = \inf_\gamma \int_\gamma \delta f_j(\zeta)\,|d\zeta|\,,$$

where γ ranges over all rectifiable curves connecting z and w in D, $R(f_j)$ becomes a metric space. In general,

$$d_j(f_j(z), f_j(w)) \leqq d(f_j(z), f_j(w); f_j)\,, \qquad z, w \in D\,.$$

If $R(f_j)$ is one-sheeted and convex in S_j in terms of the geodesics, then the equality in the above always holds. Let g_j: $D \to S_j$ be meromorphic. A mapping Φ_j: $R(f_j) \to R(g_j)$ is called isometry if Φ_j is surjective and

$$d(f_j(z), f_j(w); f_j) = d(\Phi_j \circ f_j(z), \Phi_j \circ f_j(w); g_j)$$

for $z, w \in D$. Each ϕ_j of T_j induces, in a natural manner, an isometry from $R(f_j)$ onto $R(\phi_j \circ f_j)$ or $R(\bar{\phi}_j \circ f_j)$ according as ϕ_j is conformal or anticonformal.

THEOREM j. *Let f_j: $D \to S_j$ and g_j: $D \to S_j$ be meromorphic. Each isometry Φ_j: $R(f_j) \to R(g_j)$ then has a unique ϕ_j of T_j such that $\Phi_j = \phi_j$ on $R(f_j)$.*

When does $g_j^{-1} \circ \Phi_j \circ f_j$ become identity? An example is supplied in

COROLLARY j. *If f_j: $D \to S_j$ and g_j: $D \to S_j$ are meromorphic with $\delta f_j = \delta g_j$ in D, then there exists a meromorphic $\phi_j \in T_j$ such that $g_j = \phi_j \circ f_j$.*

In particular, the map which sends $f_j(z)$ to $g_j(z)$ is an isometry. The converse of Corollary j for each j is trivial. Corollary 2 is also immediate.

3. Lemma

A key lemma is

LEMMA j. *Let Q be an open Euclidean disk contained in S_j and let Φ_j: $Q \to S_j$ be an isometry in the sense that*

$$d_j(z, w) = d_j(\Phi_j(z), \Phi_j(w))\,, \qquad z, w \in Q\,.$$

Then, Φ_j is a restriction of a $\phi_j \in T_j$ to Q.

Proof of Lemma j. Since Q itself is also a disk in S_j in the d_j metric (a d_j-disk, for short) with some d_j-center, we can find $\phi_j \in T_j$ such that $\phi_j(Q)$ is also a Euclidean disk of center zero. Furthermore, there exists $\psi_j \in T_j$ such that $\psi_j \circ \Phi_j \circ \phi_j^{-1}(0) = 0$. Thus, we may assume that Q has the Euclidean center 0 and $\Phi_j(0) = 0$.

We shall drop the suffix j if there is no confusion. There exists $p>0$ with $p\in Q$, so that $ip\in Q$. First, it follows from the definition of d_j that, for all $z\in Q$,

$$d(\Phi(z),0)=d(z,0) \qquad \text{or} \quad |\Phi(z)|=|z|\,.$$

Next, let $\Phi(p)=pe^{i\alpha}$, α real. Then, for $\Psi=e^{-i\alpha}\Phi$ we have, for all $z\in Q$,

$$|\Psi(z)|=|z| \tag{3.1}$$

and $d(\Psi(z),p)=d(z,p)$, whence,

$$\left|\frac{\Psi(z)-p}{1+(2-j)p\Psi(z)}\right|=\left|\frac{z-p}{1+(2-j)pz}\right|. \tag{3.2}$$

Let $z\neq 0, p$. Then, (3.1) shows that $\Psi(z)$ and z are on the same circle of center 0 and (3.2) shows that both $\Psi(z)$ and z lie on a Euclidean (or, more adequately, Apollonius) circle or a straight line orthogonal to the real axis. Therefore,

$$\operatorname{Re}\Psi(z)=\operatorname{Re}z \qquad \text{for all} \quad z\in Q\,. \tag{3.3}$$

Since $|\Psi(ip)|=p$ and $\operatorname{Re}\Psi(ip)=0$, there are two possibilities; $\Psi(ip)=ip$ or $-ip$. In the case $\Psi(ip)=ip$, a simple calculation yields that, for each $z\in Q$,

$$d(-i\Psi(iz),p)=d(\Psi(iz),\Psi(ip))=d(iz,ip)=d(z,p)\,,$$

and $|-\Psi(iz)|=|z|$. Therefore, again, $\operatorname{Re}(-i\Psi(iz))=\operatorname{Re}z$ for all $z\in Q$. Since Q is a disk of center 0, we can replace iz by z to obtain $\operatorname{Re}(-i\Psi(z))=\operatorname{Re}(-iz)$ for $z\in Q$, which, combined with (3.3), yields that $\Psi(z)\equiv z$. Therefore, $\Phi=e^{i\alpha}\Psi\in T$. In the case $\Psi(ip)=-ip$, we consider for $z\in Q$,

$$d(i\Psi(iz),p)=d(\Psi(iz),\Psi(ip))=d(iz,ip)=d(z,p)\,,$$

and $|i\Psi(iz)|=|z|$. Then, $\Psi(z)\equiv\bar{z}$. Therefore, $\Phi\in T$.

4. Proof of Theorem j

We shall drop the suffix j when we can avoid confusion. The map $h=g^{-1}\circ\Phi\circ f\colon D\to D$ is topological. It is not difficult to observe that each point of the set

$$E=\{a\in D;\ \delta f(a)=0 \text{ or } \delta g(h(a))=0\}$$

is isolated. If $\Phi=\phi$ in $R^*=R(f)\setminus f(E)$, then $\Phi=\phi$ on the whole $R(f)$ by continuity.

Therefore, it suffices to show that for each $f(a)\in R^*$, there exists a one-sheeted d_j-disk of center $f(a)$ contained in $R(f)$ where $\Phi=\phi_a$ for some $\phi_a\in T$. Then, by continuation to the whole R^* we know that Φ is conformal or anticonformal on R^*, and there is a unique $\phi\in T$ with $\Phi=\phi$ on R^*.

There exist one-sheeted d_j-disks V and W of centers $f(a)$ and $\Phi(f(a))$, respectively, and $V\subset R(f)$, $W\subset R(g)$. Both V and W are convex in S_j, so that

$$d(f(z),f(w))=d(f(z),f(w);f), \qquad f(z),f(w)\in V \tag{4.1}$$

and

$$d(g(z), g(w)) = d(g(z), g(w); g)\,, \qquad g(z), g(w) \in W\,. \tag{4.2}$$

By a transformation of T we may suppose that $f(a)=0$. Then we can choose an open Euclidean disk Q of center 0 contained in V whose image $\Phi(Q) \subset W$. Now, Q itself is a d_j-disk of d_j-center $0=f(a)$. Lemma j now yields the desired disk as Q with the aid of (4.1) and (4.2).

Proof of Corollary j. The map $\Phi\colon R(f_j) \to R(g_j)$ defined by $\Phi(f_j(z)) = g_j(z)$ is isometry and conformal. Hence $g_j = \phi_j \circ f_j$ is obtained.

5. Concluding remarks

We can replace D, more generally, by a Riemann surface W. One reasonable formulation in this case is the following where $H(W)$ denotes the group of all conformal and anticonformal homeomorphisms from W onto W.

Let $f_j\colon W \to S_j$ and $g_j\colon W \to S_j$ be meromorphic. Then $R(f_j)$ is isometric to $R(g_j)$ if and only if there exist $\phi_j \in T_j$ and $h \in H(W)$ such that $\phi_j \circ f \circ h = g_j$.

Note that ϕ_j is conformal if and only if h is conformal. In the case $j=2$, W must be open, and in the case $j=3$, W must be hyperbolic, or, more precisely, $W \notin O_{AB}$. In the specified case $W = \Delta \equiv \{|z|<1\}$ we know that $T_3 = H(\Delta)$. Thus, in the case $j=3$ and $W=\Delta$, we have $g_3 = \phi \circ f_3 \circ \psi$ $(\phi, \psi \in T_3)$ for $R(f_3)$ to be isometric to $R(g_3)$.

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